REFERENCES

1. CHERNOUS'KO F. L., Decomposition and suboptimal control in dynamic systems. *Prikl. Mat. Mekh.* **54**, 6, 883–893, 1990.
2. CHERNOUS'KO F. L., Decomposition and synthesis of control in dynamic systems. *Izv. Akad. Nauk SSSR, Tekh. Kibern.* **6**, 64–82, 1990.
3. KRASOVSKII N. N., *Game-Theoretical Problems on Approach of Motions*. Nauka, Moscow, 1970.
4. KRASOVSKII N. N. and SUBBOTIN A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
5. PONTRYAGIN L. S., BOLTYANSKII V. G., GAMKRELIDZE R. V. and MISHCHENKO E. F., *Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1976.
6. AKULENKO L. D. and BOLOTNIK N. N., Optimal control synthesis for transport motion of robot manipulators. *Izv. Akad. Nauk SSSR, MTT* **4**, 21–29, 1986.

Translated by Z.L.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 771–780, 1991
Printed in Great Britain.
0021–8928/91 $15.00+.00

STATIONARY AND STATIONARIZABLE REGIMES IN NORMAL STOCHASTIC DIFFERENTIAL SYSTEMS†

N. K. MOSHCHUK and I. N. SINITSYN

Moscow

(*Received* 27 *December* 1990)

Narrow-sense stationary regimes are considered for multi-dimensional non-linear systems described by Ito stochastic differential equations with Wiener processes. The conditions for the existence of stationary and stationarizable one-dimensional distributions are derived. Exact expressions are obtained for stationary distributions in some mechanical systems.

1. MANY problems of statistical dynamics of servo systems and systems with ideal stochastic holonomic and non-holonomic constraints acted upon by position conservative and non-conservative, accelerating and dissipative, gyroscopic forces and disturbances can be reduced to normal stochastic systems by augmenting the state vector [1–3]. A normal stochastic differential system (SDS) is a stochastic system whose state is described by an Ito stochastic differential equation with an appropriate initial condition

$$\mathbf{Z}^{\cdot} = \mathbf{a}(\mathbf{Z}, t) + \mathbf{b}(\mathbf{Z}, t)\mathbf{V}, \quad \mathbf{Z}(t_0) = \mathbf{Z}_0 \tag{1.1}$$

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 895–903, 1991.

Here $\mathbf{Z} \in R^k$ is the state vector (in general, augmented), $\mathbf{a} = \mathbf{a}(\mathbf{Z}, t)$ and $\mathbf{b} = \mathbf{b}(\mathbf{Z}, t)$ are respectively $k \times 1$ and $k \times l$ deterministic functions of the corresponding variables and $\mathbf{V} = \mathbf{V}(t)$ is the l-dimensional vector of independent normal white noise with mean zero and $l \times l$ intensity matrix $\boldsymbol{\nu} = \boldsymbol{\nu}(t)$, which is the time derivative of a Wiener process. The initial value $\mathbf{Z}_0$ of the state vector at time t_0 is a random variable independent of the values of the white noise $\mathbf{V}(t)$ for $t \geqslant t_0$.

SDS (1.1) is called stationary if the intensity $\boldsymbol{\nu}$ of the white noise $\mathbf{V}(t)$ is constant and the functions $\mathbf{a}$ and $\mathbf{b}$ are time-independent $\mathbf{a} = \mathbf{a}(\mathbf{Z})$, $\mathbf{b} = \mathbf{b}(\mathbf{Z})$. In this case, SDS (1.1) takes the form

$$\mathbf{Z}^{\cdot} = \mathbf{a}(\mathbf{Z}) + \mathbf{b}(\mathbf{Z})\mathbf{V}, \quad \mathbf{Z}(t_0) = \mathbf{Z}_0 \tag{1.2}$$

SDS (1.1) is called stationarizable (reducible to stationary) if there exists a smooth invertible change of the state variables and the independent argument such that in the new variables the SDS has the form (1.2).

Many authors have studied the problem of stationary regimes in multi-dimensional stationary non-linear SDS (see, e.g. [1–4] and the references there) and have derived exact expressions for one-dimensional distributions in many important practical cases. For linear SDS, exact expressions for the distributions are given in [1]. For multi-dimensional non-linear SDS, a number of approximate methods are available for determining stationary distributions (normal approximation method, method of moments, cumulant methods, orthogonal expansion methods, etc.), based on parametrization of the distributions (see, e.g. [1]).

Let us consider the problem of finding exact expressions for one-dimensional narrow-sense stationary and stationarizable distributions in SDS (1.2)

As we know (see, e.g. [1]), the one-dimensional density $f = f(\mathbf{z}, t)$ of the stochastic process $\mathbf{Z}(t)$ in SDS (1.1) satisfies the Fokker–Planck–Kolmogorov (FPK) equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \mathbf{z}}[\mathbf{a}(\mathbf{z}, t) f] + \frac{1}{2}\operatorname{tr}\left\{\frac{\partial}{\partial \mathbf{z}}\frac{\partial^T}{\partial \mathbf{z}}[\boldsymbol{\sigma}(\mathbf{z}, t) f]\right\}, \qquad \boldsymbol{\sigma} = \mathbf{b}\boldsymbol{\nu}\mathbf{b}^T \tag{1.3}$$

with appropriate initial condition and normalization condition.

2. *Proposition 1.* The function $f = f(\mathbf{z}, t)$ is the solution of the FPK equation (1.3) if and only if the vector field $\mathbf{a}(\mathbf{z}, t)$ can be represented in the form $\mathbf{a}(\mathbf{z}, t) = \mathbf{a}_1(\mathbf{z}, t) + \mathbf{a}_2(\mathbf{z}, t)$ such that the function f is the density of a finite invariant measure of the system of ordinary differential equations

$$\mathbf{z}^{\cdot} = \mathbf{a}_1(\mathbf{z}, t) \tag{2.1}$$

i.e. it satisfies the condition

$$\partial^T/\partial \mathbf{z}\,(\mathbf{a}_1 f) = 0 \tag{2.2}$$

and the component $\mathbf{a}_2 = \mathbf{a}_2(\mathbf{z}, t)$ is defined by the formula

$$\mathbf{a}_2 = [\boldsymbol{\sigma}\partial \ln f/\partial \mathbf{z} + (\partial^T/\partial \mathbf{z}\boldsymbol{\sigma})^T]/2 \tag{2.3}$$

The proof follows immediately from the FPK equation. Rewrite (1.3) in the form

$$\frac{\partial f}{\partial t} + \frac{\partial^T}{\partial \mathbf{z}} f\left\{\mathbf{a} - \frac{1}{2f}\left[\frac{\partial^T}{\partial \mathbf{z}}(\boldsymbol{\sigma} f)\right]^T\right\} = 0 \tag{2.4}$$

Hence clearly f is the solution of (1.3) if and only if

$$\partial f/\partial t + \partial^T/\partial \mathbf{z}\,(\mathbf{a}_1 f) = 0$$

where $\mathbf{a}_1 = \mathbf{a} - \mathbf{a}_2$.

Remarks. 1. The system of ordinary differential equations (2.1) is called unperturbed. This definition enables us to interpret the original SDS (1.1) as the result of perturbation of system (2.1) by some dissipative components $\mathbf{a}_2$ and random terms $\mathbf{bV}$. This leads to an analogy between SDS (1.1) and the smooth dynamic system described by Eqs (2.1). According to this analogy, the expectation of a known state function goes to the mean over a measure. The analogy may be useful for proving the non-existence of stationary distributions (see Sec. 4).

2. The unperturbed system always exists, but it is of special interest only when a stationary distribution exists. For the stationary distribution the unperturbed system is the autonomous system $\mathbf{a}_1 = \mathbf{a}_1(\mathbf{z})$.

For the stationary SDS (1.2) the unperturbed system is described by the equation

$$\mathbf{z}^{\cdot} = \mathbf{a}_1(\mathbf{z}) \tag{2.5}$$

Therefore by Proposition 1 a narrow-sense stationary solution exists if and only if the representation $\mathbf{a}(\mathbf{z}) = \mathbf{a}_1(\mathbf{z}) + \mathbf{a}_2(\mathbf{z})$ exists where the unperturbed system (2.5) has a finite invariant measure with density $f(\mathbf{z})$. Thus, stationary regimes in the original SDS (1.2) exist if (and only if) it may be treated as a perturbation of the unperturbed system (2.5) with a finite invariant measure by some dissipative and stochastic disturbances. The determination of $\mathbf{a}_1(\mathbf{z})$ and $\mathbf{a}_2(\mathbf{z})$ that satisfy the conditions of Proposition 1 is of course no less difficult than the solution of the FPK equation (1.3). However in applications (e.g. in problems of mechanics), the original vector field $\mathbf{a}(\mathbf{z})$ is often directly representable in the form $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, where the system (2.5) has an integral invariant (or even a whole family). This suggests a constructive approach to the search for one-dimensional stationary distributions.

Proposition 2. Assume that in SDS (1.2) the vector field $\mathbf{a}(\mathbf{z})$ can be represented in the form $\mathbf{a} = \mathbf{a}_1(\mathbf{z}) + \mathbf{a}_2(\mathbf{z})$ and $\mu^T/\partial\mathbf{z}(\mathbf{a}_1\mu) = 0$, where $\mu(\mathbf{z}) \geqslant 0$ is some scalar function. If

(1) the inverse matrix $\boldsymbol{\sigma}^{-1}(\mathbf{z})$ exists,

(2) the vector field $\gamma(\mathbf{z}) = \boldsymbol{\sigma}^{-1}\{2\mathbf{a}_2 - \mu^{-1}[\partial^T\}\partial\mathbf{z}(\boldsymbol{\sigma}\mu)]^T\}$ is irrotational, i.e. $\partial\gamma_i/\partial z_j = \partial\gamma_j/\partial z_i$; $i, j = 1, \ldots, k$,

(3) $F(\mathbf{z}) = \int \gamma^T(\mathbf{z})\,d\mathbf{z}$ is the first integral of system (2.5),

(4) $\int_{-\infty}^{\infty} \mu(\mathbf{z}) \exp F(\mathbf{z})\,d\mathbf{z} = 1$,

then $f(\mathbf{z}) = \mu(\mathbf{z}) \exp F(\mathbf{z})$ is the one-dimensional density of the stationary process $\mathbf{Z}(t)$ of SDS (1.2).

Proof. Represent $f(\mathbf{z})$ in the form $f = \mu f_0$ and rewrite the FPK equation in the form

$$\partial^T/\partial\mathbf{z}\,(\mathbf{a}_1\mu f_0) + \partial^T/\partial\mathbf{z}\boldsymbol{\Theta} = 0, \quad \boldsymbol{\Theta} = \mathbf{a}_2\mu f_0 - {}^1/_2\,[\partial^T/\partial\mathbf{z}\,(\boldsymbol{\sigma}\mu f_0)]^T$$

We obtain the following expression:

$$\boldsymbol{\Theta} = \mu f_0\,\{\mathbf{a}_2 - {}^1/_2\mu^{-1}\,[\partial^T/\partial\mathbf{z}\,(\boldsymbol{\sigma}\mu)]^T - {}^1/_2\boldsymbol{\sigma}\partial\ln f_0/\partial\mathbf{z}\} = {}^1/_2\mu f_0\boldsymbol{\sigma}\,[\gamma(\mathbf{z}) - \partial\ln f_0/\partial\mathbf{z}]$$

We thus see that if the field $\gamma(\mathbf{z})$ is irrotational and

$$\ln f_0 = \int \gamma^T(\mathbf{z})d\mathbf{z}, \text{ then } \boldsymbol{\Theta} = 0.$$

Now

$$\partial^T/\partial\mathbf{z}\,(\mathbf{a}_1\mu f_0) = f_0\partial^T/\partial\mathbf{z}\,(\mathbf{a}_1\mu) + \mathbf{a}_1{}^T\mu\partial f_0/\partial\mathbf{z} = \mu\mathbf{a}_1{}^T\partial f_0/\partial\mathbf{z}$$

If f_0 is the first integral of system (2.5), then $\mathbf{a}_1^T\partial f_0/\partial\mathbf{z} = 0$ and $\partial^T/\partial\mathbf{z}(\mathbf{a}_1\mu f_0) = 0$. Therefore, $\partial^T/\partial\mathbf{z}(\mathbf{a}_1\mu f_0) = \partial^T/\partial\mathbf{z}\Theta = 0$.

Remarks. 1. The density of the integral invariant $\mu(\mathbf{z})$ and the first integral $f_0(\mathbf{z})$ are defined, apart from a constant multiplier.

2. If all the conditions of Proposition 2 are satisfied, system (2.5) has a whole family of invariant measures whose density equals the product of μ by an arbitrary function of the first integrals. Clearly, $f(\mathbf{z})$ is also the density of an invariant measure of system (2.5).

3. Proposition 2 can be generalized to the case when $\boldsymbol{\Theta} \neq 0$ but $\boldsymbol{\Theta}$ = const.

Let us separately consider an important special case of SDS (1.2),

$$\mathbf{q}^{\boldsymbol{\cdot}} = \mathbf{Q}(\mathbf{q}, \mathbf{p}), \quad \mathbf{p}^{\boldsymbol{\cdot}} = \mathbf{P}(\mathbf{q}, \mathbf{p}) + \mathbf{b}(\mathbf{q})\mathbf{V} \tag{2.6}$$

which often arises in the analysis of mechanical systems exposed to random disturbances. In (2.6), $\mathbf{q} = [q_1, \ldots, q_n]^T$ is the coordinate vector, $\mathbf{p} = [p_1, \ldots, p_m]^T$ is the vector of momenta, $\mathbf{Q}$ and $\mathbf{P}$ are deterministic functions of the same dimension as the vectors $\mathbf{q}$ and $\mathbf{p}$, respectively, and $\mathbf{V}(t)$ is the vector of normal stationary white noise. For this system, the matrix $\boldsymbol{\sigma}$ in SDS (1.2) is singular.

Let us restate Proposition 2 for this special case.

Proposition 3. Assume that we have the representation of the vector $\mathbf{P}(\mathbf{p}, \mathbf{q}) = \mathbf{P}_1(\mathbf{q}, \mathbf{p}) + \mathbf{P}_2(\mathbf{q}, \mathbf{p})$, and $\partial^T/\partial\mathbf{q}(\mathbf{Q}\mu) + \partial^T/\partial\mathbf{p}(\mathbf{P}_1\mu) = 0$, $\mu(\mathbf{q}, \mathbf{p}) \geqslant 0$. Then if

(1) $\det \boldsymbol{\sigma}(\mathbf{p}) \neq 0$ ($\boldsymbol{\sigma} = \mathbf{b}\boldsymbol{\nu}\mathbf{b}^T$),

(2) $\partial\gamma_i/\partial p_j = \partial\gamma_j/\partial p_i$ $(i, j = 1, \ldots, m)$; $\boldsymbol{\gamma}(\mathbf{q}, \mathbf{p}) = 2\boldsymbol{\sigma}^{-1}(\mathbf{q})\mathbf{P}_2$,

(3) $F(\mathbf{q}, \mathbf{p}) = \int \boldsymbol{\gamma}^T(\mathbf{q}, \mathbf{p})\,d\mathbf{p}$ is the first integral of the system $\mathbf{q}^{\boldsymbol{\cdot}} = \mathbf{Q}$, $\mathbf{p}^{\boldsymbol{\cdot}} = \mathbf{P}_1$,

(4) the function $f(\mathbf{q}, \mathbf{p}) = \boldsymbol{\mu}(\mathbf{q}, \mathbf{p}) \exp F(\mathbf{q}, \mathbf{p})$ satisfies the normalization condition, then $f(\mathbf{q}, \mathbf{p})$ is the one-dimensional density of the stationary process $[\mathbf{q}(t)^T \mathbf{p}(t)^T]^T$ of the system (2.6).

Note that if $\mathbf{P}_2 = \partial\Phi(\mathbf{q}, \mathbf{p})/\partial\mathbf{p}$, then condition 2 implies that the commutator of the matrices $\boldsymbol{\sigma}^{-1}$ and $\Phi_{\mathbf{pp}} = (\partial/\partial\mathbf{p})(\partial^T/\partial\mathbf{p})\Phi$ vanishes.

Proposition 1 can also be naturally extended to the system (2.6).

Proposition 4. The function $f(\mathbf{q}, \mathbf{p})$ is the solution of Eq. (1.3) for SDS (2.6) if and only if the vector $\mathbf{P}(\mathbf{q}, \mathbf{p})$ has the representation $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ so that f is the density of a finite invariant measure of the system

$$\mathbf{q}^{\boldsymbol{\cdot}} = \mathbf{Q}, \quad \mathbf{p}^{\boldsymbol{\cdot}} = \mathbf{P}_1 \tag{2.7}$$

and the function $\mathbf{P}_2$ is defined by the formula

$$\mathbf{P}_2 = {}^1\!/_2\boldsymbol{\sigma}(\mathbf{q})\,\partial \ln f/\partial\mathbf{p} \tag{2.8}$$

The problem of determining the one-dimensional stationary distributions in system (2.6) is thus restated as the problem of determining the density f of an invariant measure (if it exists) of a system of special form

$$\mathbf{q}^{\boldsymbol{\cdot}} = \mathbf{Q}(\mathbf{q}, \mathbf{p}), \quad \mathbf{p}^{\boldsymbol{\cdot}} = \mathbf{P}(\mathbf{q}, \mathbf{p}) - {}^1\!/_2\boldsymbol{\sigma}(\mathbf{q})\,\partial \ln f/\partial\mathbf{p} \tag{2.9}$$

Stochastic perturbation of a system of ordinary differential equations with a finite invariant measure has been considered in [4], where sufficient conditions have been obtained for the existence of narrow-sense stationary solutions whose one-dimensional density is simply identical with the density of the invariant measure of the unperturbed system. In fact, these conditions are also necessary. We will give these conditions below.

Consider the SDS

$$\mathbf{Z}^{\boldsymbol{\cdot}} = \mathbf{a}_1(\mathbf{Z}) + \mathbf{a}_2(\mathbf{Z}) + \mathbf{b}(\mathbf{Z})\mathbf{V} \tag{2.10}$$

Assume that the scalar function $N(\mathbf{z})$ is the density of a finite invariant measure of system $\mathbf{z}^{\bullet} = \mathbf{a}_1(\mathbf{z})$, i.e.

$$\frac{\partial^T}{\partial \mathbf{z}}(\mathbf{a}_1 N) = 0, \quad N \geqslant 0, \quad \int_{-\infty}^{\infty} N\, d\mathbf{z} = 1$$

Proposition 5. The function $N(\mathbf{z})$ is a one-dimensional density of the stationary process of SDS (2.10) if and only if there exists a matrix function $\mathbf{A}(\mathbf{z})$ such that

$$1)\ \mathbf{a}_2 N = (\partial^T/\partial \mathbf{z}\mathbf{A})^T, \qquad 2)\ \mathbf{A} + \mathbf{A}^T = \sigma N \tag{2.11}$$

Let us compare propositions 2, 3, and 5. In all these propositions, the required one-dimensional density is identical with the density of the invariant measure of the unperturbed system. However, propositions 2 and 3 are best used in cases when the first integrals of the unperturbed system are not known in advance and we only know the density of an invariant (not necessarily finite) measure. If for the unperturbed system we know both the density of the invariant measure and some (or all) first integrals, it is preferable to use Proposition 5.

For example, consider the natural mechanical system $\mathbf{q}^{\bullet} = \mathbf{Q}(\mathbf{q}, \mathbf{p})$, $\mathbf{p}^{\bullet} = \mathbf{P}(\mathbf{q}, \mathbf{p})$ which has an invariant measure with density $N(\mathbf{q})$ and a collection of first integrals $H_1, \ldots, H_\varkappa$ that are quadratic or linear in $\mathbf{p}$. If this system is situated in a random environment, the one-dimensional stationary density can be sought in the form

$$f(\mathbf{q}, \mathbf{p}) = N(\mathbf{q}) \exp(-\theta H) \tag{2.12}$$

where θ is a scalar parameter, and the function H can be identified with the sum of Chetayev integrals

$$H = \sum \lambda_i H_i + \sum \lambda_j' H_j^2$$

Here λ_i, λ_j' are some constants, chosen so that the function H is a positive definite quadratic form of the momenta.

Proposition 5 has been used to derive [4] the conditions for the existence of stationary modes in stochastic Chaplygin non-holonomic systems with stochastic instability of the constraints.

An important practical case in mechanics is the case of stationary distributions for which the logarithm of the one-dimensional density is a quadratic form in some of the variables.

Consider the Hamiltonian system with the Hamiltonian $H = \mathbf{p}^{T\Omega}(\mathbf{q})\mathbf{p}/2 + \Pi(\mathbf{q})$ acted upon by dissipative forces linear in the momenta and also by random forces

$$\mathbf{q}^{\bullet} = \partial H/\partial \mathbf{p}, \quad \mathbf{p}^{\bullet} = -\partial H/\partial \mathbf{q} + \mathbf{D}(\mathbf{q})\mathbf{p} + \mathbf{b}(\mathbf{q})\mathbf{V} \tag{2.13}$$

Here $\mathbf{V}(t)$ is the vector of stationary normal white noise.

Let us find the necessary and sufficient conditions for the existence of stationary distributions with one-dimensional density of the form $f = c\exp(-F)$, $F = \mathbf{p}^T\mathbf{A}(\mathbf{q})\mathbf{p}/2 + h(\mathbf{1})$ is a positive-definite form of $\mathbf{p}$. These distributions are normal with respect to the momenta $\mathbf{p}$.

By direct substitution into the FPK equation we can prove the following.

Proposition 6. The function $f(\mathbf{q}, \mathbf{p})$ is the solution of the FPK equation if and only if the following conditions are satisfied:

(1) $\{F, H\} = 0$, 2) $\mathbf{AD} + \mathbf{D}^T\mathbf{A} + \mathbf{A}\sigma\mathbf{A} = 0$

This proposition can be generalized to the case of SDS of the form

$$\mathbf{q}^{\bullet} = \mathbf{Q}(\mathbf{q}, \mathbf{p}), \quad \mathbf{p}^{\bullet} = \mathbf{P}(\mathbf{q}, \mathbf{p}) + \mathbf{D}(\mathbf{q})\mathbf{p} + \mathbf{b}(\mathbf{q})\mathbf{V}$$

Here $\mathbf{Q}$ is a linear function and the function $\mathbf{P}$ is quadratic in the momenta $\mathbf{p}$. Such systems describe for instance, stochastic Chaplygin non-holonomic systems [2].

Assume that the equations $\mathbf{q}^{\cdot} = \mathbf{Q}$, $\mathbf{p}^{\cdot} = \mathbf{P}$ have an integral invariant with density $\mu(\mathbf{q}) \geqslant 0$.

Proposition 7. The function $f = c\exp(-F)$ is the solution of the FPK equation (1.3) if and only if the following conditions are satisfied:

(1) $F - \ln\mu$ is the first integral of the equations $\mathbf{q}^{\cdot} = \mathbf{Q}$, $\mathbf{p}^{\cdot} = \mathbf{P}$;
(2) $\mathbf{AD} + \mathbf{D}^T\mathbf{A} + \mathbf{A}\sigma\mathbf{A} = 0$

3. Consider the forced motion of a system of material points of mass m_s ($s = 1, 2, \ldots$) relative to some Cartesian coordinates. The position of the system is defined by the radii-vectors $\mathbf{r}_s$ of its points. The constraints are restraining, holonomic, stationary, and ideal. Denote by $\mathbf{q} = [q_1, \ldots, q_n]^T$ the vector of generalized coordinates of the system. Assume that the system is in a homogeneous field of random forces with acceleration represented by a vector of independent normal white noise $\mathbf{V}$ of constant intensity ν and also in a field of potential and dissipative forces with dissipation function Φ proportional to the square of the velocity of the centre of mass.

Let us write the equations of motion in Hamiltonian form. The vector $\mathbf{Q}_1$ of generalized random forces is $\mathbf{Q}_1 = \mathbf{b}(\mathbf{q})\mathbf{V}$, $\mathbf{b}(\mathbf{q}) = \partial/\partial\mathbf{q}\,(M\mathbf{r})$. Here M is the mass of the entire system and

$$\mathbf{r} = \sum_s m_s \mathbf{r}_s / M$$

is the radius-vector of the centre of mass of the system. Indeed,

$$\sum_s m_s \mathbf{V}\delta\mathbf{r}_s = \sum_s m_s \mathbf{V} \sum_{i=1}^{n} \frac{\partial \mathbf{r}_s}{\partial q_i}\delta q_i = M \sum_{i=1}^{n} \frac{\partial \mathbf{r}}{\partial q_i} \mathbf{V}\delta q_i$$

Since

$$\Phi = \varepsilon\,(M\mathbf{r}^{\cdot})^2/2 \quad (\varepsilon = \text{const}),$$

the vector $\mathbf{Q}_2$ of generalized resistance forces is given by

$$\mathbf{Q}_2 = -\varepsilon\mathbf{a}(\mathbf{q})\,\mathbf{q}^{\cdot}, \quad \mathbf{a}(\mathbf{q}) = \mathbf{b}\mathbf{b}^T$$

In fact

$$-\mathbf{Q}_2 = \frac{\partial}{\partial \mathbf{q}^{\cdot}}\left[\frac{1}{2}\varepsilon\,(M\mathbf{r}^{\cdot})^2\right] = \frac{1}{2}\varepsilon\frac{\partial}{\partial \mathbf{q}^{\cdot}}\left(\frac{\partial^T M\mathbf{r}}{\partial \mathbf{q}}\mathbf{q}^{\cdot}\right)^2 = \varepsilon\frac{\partial M\mathbf{r}}{\partial \mathbf{q}}\frac{\partial^T M\mathbf{r}}{\partial \mathbf{q}}\mathbf{q}^{\cdot}$$

Therefore, the stochastic equations of motion of the system have the form

$$\mathbf{q}^{\cdot} = \frac{\partial H}{\partial \mathbf{p}}, \quad \mathbf{p}^{\cdot} = -\frac{\partial H}{\partial \mathbf{q}} - \varepsilon\mathbf{a}(\mathbf{q})\frac{\partial H}{\partial \mathbf{p}} + \mathbf{b}(\mathbf{q})\,\mathbf{V} \tag{3.1}$$

where H is the system Hamiltonian.

We will now use Proposition 3. In this case $\mathbf{P}_1 = -\partial H/\partial\mathbf{q}$, $\mathbf{P}_2 = \mathbf{Q}_2$, $\mu = 1$, $\det\boldsymbol{\sigma}(\mathbf{q}) \neq 0$ ($\boldsymbol{\sigma} = \boldsymbol{\gamma}\mathbf{b}\mathbf{b}^T$), and the vector field

$$\boldsymbol{\gamma}(\mathbf{q}, \mathbf{p}) = -2\sigma^{-1}\varepsilon\mathbf{a}\partial H/\partial\mathbf{p} = -2\varepsilon\nu^{-1}\partial H/\partial\mathbf{p} \tag{3.2}$$

From condition 3 we obtain $F = -2\varepsilon\nu^{-1}H$.

Thus, if the function

$$f(\mathbf{q}, \mathbf{p}) = c \exp[-2\varepsilon\nu^{-1}H(\mathbf{q}, \mathbf{p})] \tag{3.3}$$

satisfies the normalization condition, it is a one-dimensional density of a narrow-sense stationary solution.

Note that the homogeneous field of random forces arises if the system is in a state of translational vibration with a white noise acceleration vector.

Consider an n-link planar pendulum that moves in a homogeneous gravitational field and its suspension point vibrates with accelerations that are independent normal white noise. If the dissipative function is proportional to the square of the velocity (relative to the suspension point) of the centre of mass, then there exists a narrow-sense stationary solution whose one-dimensional density is given by the Gibbs formula (3.3).

Note that the imposition of some new holonomic constraint on the system (3.1) does not affect the existence of a stationary distribution. The form of the function H and of the matrices $\mathbf{a}$ and $\boldsymbol{\sigma}$ changes, but formulas (3.2) and (3.3) remain valid.

A more more interesting question is the effect of Chaplygin non-holonomic constraints on the original system (3.1). Let $\mathbf{q} = [\mathbf{q}'^T\mathbf{q}''^T]^T$ and assume that the non-holonomic constraint equations have the form ($\boldsymbol{\Lambda}$ is some matrix)

$$\mathbf{q}'^{\boldsymbol{\cdot}} = \boldsymbol{\Lambda}(\mathbf{q}'')\mathbf{q}''^{\boldsymbol{\cdot}}$$

Then equations (3.1) should be replaced with

$$\mathbf{q}''^{\boldsymbol{\cdot}} = \frac{\partial H^*}{\partial \mathbf{p}''}, \quad \mathbf{p}''^{\boldsymbol{\cdot}} = -\frac{\partial H^*}{\partial \mathbf{q}''} + \boldsymbol{\Gamma}(\mathbf{q}'', \mathbf{p}'') - \varepsilon \mathbf{a}^*(\mathbf{q}'')\frac{\partial H^*}{\partial \mathbf{p}''} + b^*(\mathbf{q}'')\mathbf{V} \tag{3.4}$$

where $H^*(\mathbf{q}'', \mathbf{p}'')$ is the Hamiltonian and $\boldsymbol{\Gamma}$ is the column vector of non-holonomic terms. Now $\mathbf{P}_1 = -\partial H/\partial \mathbf{q}'' + \boldsymbol{\Gamma}$. Computations show that $\gamma(\mathbf{q}'', \mathbf{p}')$ is determined by expression (3.2) where H and $\mathbf{p}$ are replaced with H^* and $\mathbf{p}''$. However, the existence of a stationary regime in this case depends on the existence of an invariant measure of the system (3.4) for $\varepsilon = 0$, $\mathbf{V} = 0$. If it exists and its density is $\mu(\mathbf{q}'')$, then a stationary regime exists and its one-dimensional density is given by (3.3) with c replaced by $\mu(\mathbf{q}'')$.

Thus, when Chaplygin non-holonomic constraints are imposed on system (3.1), the stationary regime persists if the resulting non-holonomic system (without dissipation and random disturbances) has an invariant measure.

4. As we have noted above, the problem of finding the stationary distributions reduces to the problem of finding the density f of an invariant measure of the unperturbed system. This is of course difficult because the right-hand side of the equations of the unperturbed system contains the unknown function f. However, the analogy may be useful when proving the non-existence of stationary distributions, because a necessary condition for the existence of stationary distributions is the existence of some invariant measure of the unperturbed system.

Let us demonstrate this with an example of a smooth system (2.6) that corresponds to the unperturbed system (2.9). Assume that Eqs (2.9) have a singular point for any function f. Without loss of generality, we assume that this point is $\mathbf{p} = 0$, $\mathbf{q} = 0$. This is so if $\mathbf{b}(0) = 0$, $\mathbf{P}(0, 0) = 0$, and $\mathbf{Q}(0, 0) = 0$. Then the relationship

$$\left(\frac{\partial^T}{\partial \mathbf{q}}\mathbf{Q} + \frac{\partial^T}{\partial \mathbf{p}}\mathbf{P}\right)_{\mathbf{q}=0,\,\mathbf{p}=0} = 0 \tag{4.1}$$

will be the necessary condition for the existence of an invariant measure (and therefore of a stationary distribution) with a smooth density for the system (2.9) in the neighbourhood of equilibrium [5].

As an example, consider the plane motion of a mathematical pendulum of unit length in a homogeneous gravitational field. We assume that the point of suspension vibrates vertically with random acceleration V, which is normal white noise of constant intensity. The pendulum is acted upon by a dissipative moment proportional to the angular velocity. The equations of motion have the form

$$\varphi^\cdot = \omega, \quad \omega^\cdot = -g \sin \varphi - d\omega + V \sin \varphi$$

where $d>0$ is the coefficient of friction and g is the acceleration due to gravity. The necessary condition (4.1) is not satisfied in this case:

$$\left[\frac{\partial}{\partial\varphi}\omega + \frac{\partial}{\partial\omega}(-g \sin \varphi - d\omega)\right]_{\varphi=0,\ \omega=0} = -d \neq 0$$

Therefore, this system does not have a stationary distribution with a smooth one-dimensional density. Such a distribution may exist if, for instance, $d = d(\varphi)$ and $d(0) = 0$. Thus, singularity of the diffusion matrix at singular points of the equations of the system without random disturbances may be an obstacle to the existence of stationary distributions with a smooth one-dimensional density.

5. In the dynamics of holonomic systems with random disturbances of the type of normal white noise, we can sometimes find a Gibbs stationary distribution with one-dimensional density of the form [1]

$$f(\mathbf{q}, \mathbf{p}) = c \exp[-\theta H(\mathbf{q}, \mathbf{p})], \quad c, \theta = \text{const} \tag{5.1}$$

where H is the Hamiltonian of some system. We will discuss two interesting properties of this distribution associated with the variation of stiffness and mass in the system. Consider (5.1) for the case when H is the Hamiltonian of a natural mechanical system with n degrees of freedom, i.e. $H = \mathbf{p}^T\mathbf{\Omega}\mathbf{p}/2 + \Pi(\mathbf{q})$, and $\mathbf{\Omega}$ is a constant matrix. In this case the expectation of $\mathbf{p}$ is zero ($M\mathbf{p} = 0$) and the normalizing constant c can be calculated explicitly. We thus obtain

$$f(\mathbf{q}, \mathbf{p}) = \sqrt{\frac{|\theta\Omega|}{(2\pi)^n}} \exp(-\theta H) \left[\int_{-\infty}^{\infty} \exp(-\theta\Pi)\, d\mathbf{q}\right]^{-1} \tag{5.2}$$

Let us compare two systems with different matrices $\mathbf{\Omega}$. We say that the system with $\mathbf{\Omega}_2$ has a greater mass than the system with $\mathbf{\Omega}_1$ if $\mathbf{p}^T\mathbf{\Omega}_1\mathbf{p} \geq \mathbf{p}^T\mathbf{\Omega}_2\mathbf{p}$ (or equivalently $\mathbf{q}^{\cdot T}\mathbf{\Omega}_1\mathbf{q}^\cdot \leq \mathbf{q}^{\cdot T}\Delta_2\mathbf{q}^\cdot$). We will first prove that the variances of the momenta Dp_i ($i = 1, \ldots, n$) increase as the system mass increases (first property).

Indeed, we have

$$Dp_i = \sqrt{\frac{|\theta\Omega|}{(2\pi)^n}} \int_{-\infty}^{\infty} p_i^2 \exp\left(-\frac{1}{2}\theta\mathbf{p}^T\mathbf{\Omega}\mathbf{p}\right) d\mathbf{p} = k_{ii} \tag{5.3}$$

where k_{ii} is the diagonal element of the matrix $K = \theta^{-1}\mathbf{\Omega}^{-1}$. Since the second system is stiffer, the ellipsoid $\mathbf{p}^T\mathbf{\Omega}_2\mathbf{p} = \xi = \text{const}$ is inside the ellipsoid $\mathbf{p}^T\mathbf{\Omega}_1\mathbf{p} = \xi$, and any line drawn from the origin therefore intersects first the second ellipsoid and only then the first ellipsoid. Hence we obtain

$$1/\sqrt{k_{ii}^{(2)}} \leq 1/\sqrt{k_{ii}^{(1)}} \quad \text{and} \quad k_{ii}^{(2)} \geq k_{ii}^{(1)} \tag{5.4}$$

Here the superscript is the system index.

Note that if we additionally stipulate that $\Pi(-\mathbf{q}) = \Pi(\mathbf{q})$, i.e. the expectation of $\mathbf{q}$ is zero, then we can show that Dq_i ($i = 1, \ldots, n$) is not affected by changes in the system mass.

Let us consider the second property. Let

$$H = \mathbf{p}^T \boldsymbol{\Omega}(\mathbf{q})\,\mathbf{p}/2 + \mathbf{q}^T \mathbf{B} \mathbf{q}/2,$$

where $\mathbf{B}$ is a constant matrix. We say that a system with potential energy matrix $\mathbf{B}_2$ is stiffer than the system with the matrix $\mathbf{B}_1$ if $\mathbf{q}^T\mathbf{B}_1\mathbf{q} \leqslant \mathbf{q}^T\mathbf{B}_2\mathbf{q}$. We can similarly show that the variances Dq_i diminish as the stiffness increases.

6. Let us consider the non-stationary processes that can be reduced to stationary processes ("stationarizable" processes) [1, 6]. We will give a rigorous formulation. A non-stationary distribution in SDS (1.1) is called reducible to a narrow-sense stationary distribution (stationarizable) if there exists a smooth invertible change of the variables and the independent argument

$$\mathbf{Y}(t) = \boldsymbol{\Psi}(\mathbf{Z}(\delta(t)),\ t) \tag{6.1}$$

such that in the new variables $\mathbf{Y}$ the SDS takes the form

$$\mathbf{Y}^{\boldsymbol{\cdot}} = \mathbf{a}^*(\mathbf{Y}) + \mathbf{b}^*(\mathbf{Y})\,\mathbf{V} \tag{6.2}$$

and has a narrow-sense stationary distribution with the density $F_1(\mathbf{y})$. Reducibility in this definition is in the narrow sense (see [1]).

We can directly verify the following proposition.

Proposition 8. SDS (1.1) has a stationarizable non-stationary distribution if and only if there exist matrix functions $\boldsymbol{\Psi}(\mathbf{Z}, t)$, $\mathbf{a}^*(\mathbf{Y})$ $(\dim \mathbf{a}^* = \dim \mathbf{Z} = k)$, $\mathbf{b}^*(\mathbf{Y})$ $(\dim \mathbf{b}^* = \dim \mathbf{b})$ and also a scalar function $\lambda(t) > 0$ such that

(1) $\det\,[\partial\boldsymbol{\Psi}/\partial\mathbf{z}] \neq 0$

(2) $\boldsymbol{\Psi}_t + \boldsymbol{\Psi}_{\mathbf{Z}}{}^T\mathbf{a} + \frac{1}{2}\boldsymbol{\Psi}_{\mathbf{ZZ}} : \boldsymbol{\sigma}\,|_{\mathbf{Z}\to\boldsymbol{\Psi}^{-1}(\mathbf{Y},t)} = \lambda(t)\,\mathbf{a}^*(\mathbf{Y})$

$\boldsymbol{\Psi}_{\mathbf{Z}}{}^T\mathbf{b}\,|_{\mathbf{Z}\to\boldsymbol{\Psi}^{-1}(\mathbf{Y},t)} = \sqrt{\lambda(t)}\,\mathbf{b}^*(\mathbf{Y})$

(3) the SDS $\mathbf{Y}^{\boldsymbol{\cdot}} = \mathbf{a}^*(\mathbf{Y}) + \mathbf{b}^*(\mathbf{Y})\mathbf{V}$ has a stationary distribution with the density $f_1(\mathbf{y})$.

The required distribution has the form

$$f(\mathbf{z},\ t) = f_1(\boldsymbol{\Psi}(\mathbf{z},\ t))\ |\,\partial\boldsymbol{\Psi}(\mathbf{z},\ t)/\partial\mathbf{z}\,| \tag{6.3}$$

In condition 2,

$$\boldsymbol{\Psi}_{\mathbf{ZZ}} : \boldsymbol{\sigma} = \begin{Vmatrix} \operatorname{tr}(\boldsymbol{\Psi}_{1\mathbf{ZZ}}\boldsymbol{\sigma}) \\ \cdots \\ \operatorname{tr}(\boldsymbol{\Psi}_{k\mathbf{ZZ}}\boldsymbol{\sigma}) \end{Vmatrix}$$

The case when stationarization is by a linear transformation is particularly interesting.

Proposition 9. SDS (1.1) has a stationarizable non-stationary distribution if there exist matrix functions $\boldsymbol{\alpha}(t)$ $(|\boldsymbol{\alpha}| \neq 0)$, $\boldsymbol{\beta}(t)$, $\mathbf{a}^*(\mathbf{Y})$, $\mathbf{b}^*(\mathbf{Y})$ and also a scalar function $\lambda(t) > 0$ such that

(1) $\boldsymbol{\alpha}^{\boldsymbol{\cdot}}\boldsymbol{\alpha}^{-1}\mathbf{Y} - \boldsymbol{\alpha}^{\boldsymbol{\cdot}}\boldsymbol{\alpha}^{-1}\boldsymbol{\beta} + \boldsymbol{\beta}^{\boldsymbol{\cdot}} + \boldsymbol{\alpha}\mathbf{a}\,(\boldsymbol{\alpha}^{-1}(\mathbf{Y} - \boldsymbol{\beta}),\ t) = \lambda\mathbf{a}^*(\mathbf{Y})$,

(2) $\boldsymbol{\alpha}\mathbf{b}\,(\boldsymbol{\alpha}^{-1}(\mathbf{Y} - \boldsymbol{\beta}),\ t) = \sqrt{\lambda}\mathbf{b}^*(\mathbf{Y})$,

(3) the SDS $\mathbf{Y}^{\boldsymbol{\cdot}} = \mathbf{a}^*(\mathbf{Y}) + \mathbf{b}^*(\mathbf{Y})\mathbf{V}$ has a stationary distribution with density $f_1(\mathbf{y})$. Then SDS (1.1) has a non-stationary distribution with density

$$f(\mathbf{z},\ t) = f_1(\boldsymbol{\alpha}\mathbf{z} + \boldsymbol{\beta},\ t)\ |\,\boldsymbol{\alpha}\,| \tag{6.4}$$

Example. Consider the one-dimensional equation (1.1). Let $a = a(Z - ct)$, $b = b(Z - ct)$ $(c = \text{const})$. Then conditions 1 and 2 of Proposition 9 hold if we take $a^* = v(Y) - c$, $b^* = b(Y)$, $\lambda = 1$. The conditions when the one-dimensional equation (6.2) has a stationary distribution and the explicit form of the density are well known (see, e.g. [1]). In these cases, the original equation has a non-stationary distribution with density $f(z-ct)$ (a "soliton" distribution). Note that such systems arise in the dynamics of variable-mass stochastic systems.

REFERENCES

1. PUGACHEV V. S. and SINITSYN I. N., *Stochastic Differential Systems: Analysis and Filtering*. Nauka, Moscow, 1990.
2. MOSHCHUK N. K. and SINITSYN I. N., On stochastic nonholonomic systems. *Prikl. Mat. Mekh.* **54**, 2, 213–223, 1990.
3. DIMENTBERG M. F., *Stochastic Processes in Dynamic Systems with Variable Parameters*. Nauka, Moscow, 1989.
4. MOSHCHUK N. K. and SINITSYN I. N., On stationary distributions in nonlinear stochastic differential systems. *Q. J. Mech. Appl. Math.* 1991.
5. KOZLOV V. V., On the existence of an integral invariant of smooth dynamic systems. *Prikl. Mat. Mekh.* **51**, 4, 538–545, 1987.
6. PUGACHEV, V. V., *The Theory of Random Functions and Its Application to Automatic Control Problems*. Fizmatgiz, Moscow, 1960

Translated by Z.L.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 780–788, 1991
Printed in Great Britain.
0021–8928/91 $15.00+.00

THE STABILITY OF A CLASS OF REVERSIBLE SYSTEMS†

A. L. KUNITSYN and M. V. MATVEYEV

Moscow

(*Received* 25 *November* 1990)

The problem of the stability of the point of rest of an autonomous system of ordinary differential equations from a class of reversible systems [1] characterized by the critical case of m zero roots and n pairs of pure imaginary roots is considered. When there are no internal resonances [2, 3], the point of rest always has Birkhoff complete stability [2]. Internal resonances may lead to Lyapunov instability. The conditions of stability and instability of the model system when there are third-order resonances may be obtained from a criterion previously developed [4] for the case of pure imaginary roots. The results are used to analyse the stability of the translational–rotational motion of an active artificial satellite in a non-Keplerian circular orbit, including a geostationary satellite in any latitude [4, 5]. The region of stability of relative equilibria and regular precession of the satellite is constructed assuming a central gravitational field and the resonance modes are analysed.

1. CONSIDER the system of equations of perturbed motion

$$\mathbf{X}^{\cdot} = D\mathbf{X} + \Phi(\mathbf{X});\quad \mathbf{X} \in R^N;\quad \Phi(0) = 0 \tag{1.1}$$

† *Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 904–911, 1991.